7-2 Inverse Functions and Relations

Main Ideas

- Find the inverse of a function or relation.
- Determine whether two functions or relations are inverses.

New Vocabulary

inverse relation
inverse function
identity function
one-to-one

GET READY for the Lesson

Most scientific formulas involve measurements given in SI (International System) units. The SI units for speed are meters per second. However, the United States uses customary measurements such as miles per hour.

To convert x miles per hour to an approximate equivalent in meters per second, you can evaluate the following.

$$f(x) = \frac{x \text{ miles}}{1 \text{ hour}} \cdot \frac{1600 \text{ meters}}{1 \text{ mile}} \cdot \frac{1 \text{ hour}}{3600 \text{ seconds}} \text{ or } f(x) = \frac{4}{9}x$$

To convert x meters per second to an approximate equivalent in miles per hour, you can evaluate the following.

$$g(x) = \frac{x \text{ meters}}{1 \text{ second}} \cdot \frac{3600 \text{ seconds}}{1 \text{ hour}} \cdot \frac{1 \text{ mile}}{1600 \text{ meters}} \text{ or } g(x) = \frac{9}{4}x$$

Notice that $f(x)$ multiplies a number by 4 and divides it by 9. The function $g(x)$ does the inverse operation of $f(x)$. It divides a number by 4 and multiplies it by 9. These functions are inverses.

Find Inverses Recall that a relation is a set of ordered pairs. The **inverse relation** is the set of ordered pairs obtained by reversing the coordinates of each ordered pair. The domain of a relation becomes the range of the inverse, and the range of a relation becomes the domain of the inverse.

KEY CONCEPT — *Inverse Relations*

Words Two relations are inverse relations if and only if whenever one relation contains the element (a, b), the other relation contains the element (b, a).

Examples $Q = \{(1, 2), (3, 4), (5, 6)\}$ $\quad S = \{(2, 1), (4, 3), (6, 5)\}$
Q and S are inverse relations.

EXAMPLE 1 Find an Inverse Relation

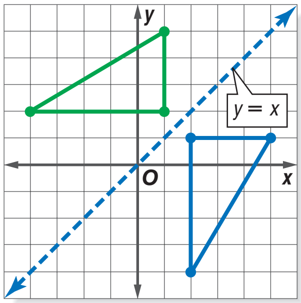
GEOMETRY The ordered pairs of the relation {(2, 1), (5, 1), (2, −4)} are the coordinates of the vertices of a right triangle. Find the inverse of this relation and determine whether the resulting ordered pairs are also the vertices of a right triangle.

To find the inverse of this relation, reverse the coordinates of the ordered pairs.

(continued on the next page)

The inverse of the relation is {(1, 2), (1, 5), (−4, 2)}.

Plotting the points shows that the ordered pairs also describe the vertices of a right triangle. Notice that the graphs of the relation and the inverse relation are reflections over the graph of $y = x$.



CHECK Your Progress

1. The ordered pairs of the relation {(−8, −3), (−8, −6), (−3, −6)} are the coordinates of the vertices of a right triangle. Find the inverse of this relation and determine whether the resulting ordered pairs are also the vertices of a right triangle.

Reading Math

f^{-1} is read *f inverse* or the *inverse of f*. Note that −1 is not an exponent.

The ordered pairs of **inverse functions** are also related. We can write the inverse of function $f(x)$ as $f^{-1}(x)$.

KEY CONCEPT ***Property of Inverse Functions***

Suppose f and f^{-1} are inverse functions. Then, $f(a) = b$ if and only if $f^{-1}(b) = a$.

Let's look at the inverse functions $f(x) = x + 2$ and $f^{-1}(x) = x - 2$.

Evaluate $f(5)$.	Now, evaluate $f^{-1}(7)$.
$f(x) = x + 2$	$f^{-1}(x) = x - 2$
$f(5) = 5 + 2$ or 7	$f^{-1}(7) = 7 - 2$ or 5

Since $f(x)$ and $f^{-1}(x)$ are inverses, $f(5) = 7$ and $f^{-1}(7) = 5$. The inverse function can be found by exchanging the domain and range of the function.

EXAMPLE 2 Find and Graph an Inverse Function

a. Find the inverse of $f(x) = \frac{x + 6}{2}$.

Step 1 Replace $f(x)$ with y in the original equation.

$$f(x) = \frac{x+6}{2} \qquad y = \frac{x+6}{2}$$

Step 2 Interchange x and y.

$$x = \frac{y+6}{2}$$

Step 3 Solve for y.

$x = \frac{y+6}{2}$ Inverse

$2x = y + 6$ Multiply each side by 2.

$2x - 6 = y$ Subtract 6 from each side.

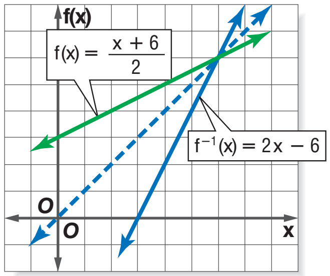
Step 4 Replace y with $f^{-1}(x)$.

$$y = 2x - 6 \qquad f^{-1}(x) = 2x - 6$$

The inverse of $f(x) = \frac{x+6}{2}$ is $f^{-1}(x) = 2x - 6$.

b. Graph the function and its inverse.

Graph both functions on the coordinate plane. The graph of $f^{-1}(x) = 2x - 6$ is the reflection of the graph of $f(x) = \frac{x+6}{2}$ over the line $y = x$.



CHECK Your Progress

2A. Find the inverse of $f(x) = \frac{x-3}{5}$.

2B. Graph the function and its inverse.

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Inverses of Relations and Functions You can determine whether two functions are inverses by finding both of their compositions. If both equal the **identity function** $I(x) = x$, then the functions are inverse functions.

KEY CONCEPT — *Inverse Functions*

Words Two functions f and g are inverse functions if and only if both of their compositions are the identity function.

Symbols $[f \circ g](x) = x$ and $[g \circ f](x) = x$

Study Tip

Inverse Functions

Both compositions of $f(x)$ and $g(x)$ must be the identity function for $f(x)$ and $g(x)$ to be inverses. It is necessary to check them both.

EXAMPLE Verify that Two Functions are Inverses

3 **Determine whether $f(x) = 5x + 10$ and $g(x) = \frac{1}{5}x - 2$ are inverse functions.**

Check to see if the compositions of $f(x)$ and $g(x)$ are identity functions.

$$\begin{aligned}[f \circ g](x) &= f[g(x)] \\ &= f\left(\tfrac{1}{5}x - 2\right) \\ &= 5\left(\tfrac{1}{5}x - 2\right) + 10 \\ &= x - 10 + 10 \\ &= x\end{aligned}$$

$$\begin{aligned}[g \circ f](x) &= g[f(x)] \\ &= g(5x + 10) \\ &= \tfrac{1}{5}(5x + 10) - 2 \\ &= x + 2 - 2 \\ &= x\end{aligned}$$

The functions are inverses since both $[f \circ g](x)$ and $[g \circ f](x)$ equal x.

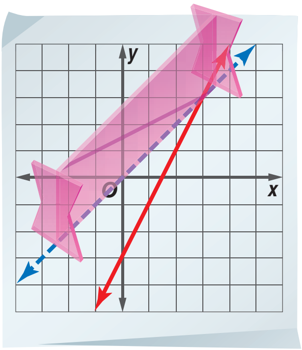
CHECK Your Progress

3. Determine whether $f(x) = 3x - 3$ and $g(x) = \frac{1}{3}x + 4$ are inverse functions.

You can also determine whether two functions are inverse functions by graphing. The graphs of a function and its inverse are mirror images with respect to the graph of the identity function $I(x) = x$.

ALGEBRA LAB

Inverses of Functions



- Use a full sheet of grid paper. Draw and label the *x*- and *y*-axes.
- Graph $y = 2x - 3$. Then graph $y = x$ as a dashed line.
- Place a geomirror so that the drawing edge is on the line $y = x$. Carefully plot the points that are part of the reflection of the original line. Draw a line through the points.

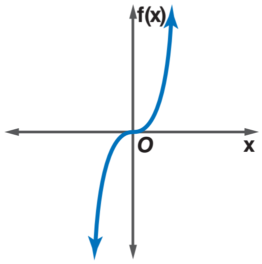
ANALYZE

1. What is the equation of the drawn line?

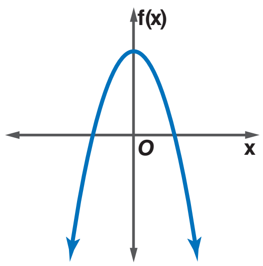
2. What is the relationship between the line $y = 2x - 3$ and the line that you drew? Justify your answer.

3. Try this activity with the function $y = |x|$. Is the inverse also a function? Explain.

When the inverse of a function is a function, then the original function is said to be **one-to-one**. Recall that the *vertical line test* can be used to determine if a graph represents a function. Similarly, the *horizontal line test* can be used to determine if the inverse of a function is a function.



No horizontal line can be drawn so that it passes through more than point. The inverse of this function is a function.



A horizontal line can be drawn so that it passes through more than one point. The inverse of this function is not a function.

CHECK Your Understanding

Example 1 (pp. 391–392)

Find the inverse of each relation.

1. $\{(2, 4), (-3, 1), (2, 8)\}$

2. $\{(1, 3), (1, -1), (1, -3), (1, 1)\}$

Example 2 (pp. 392–393)

Find the inverse of each function. Then graph the function and its inverse.

3. $f(x) = -x$

4. $g(x) = 3x + 1$

5. $y = \frac{1}{2}x + 5$

PHYSICS For Exercises 6 and 7, use the following information.
The acceleration due to gravity is 9.8 meters per second squared (m/s^2). To convert to feet per second squared, you can use the following operations.

$$\frac{9.8\ \text{m}}{\text{s}^2} \times \frac{100\ \text{cm}}{1\ \text{m}} \times \frac{1\ \text{in.}}{2.54\ \text{cm}} \times \frac{1\ \text{ft}}{12\ \text{in.}}$$

6. Find the value of the acceleration due to gravity in feet per second squared.

7. An object is accelerating at 50 feet per second squared. How fast is it accelerating in meters per second squared?

Example 3 (p. 393)

Determine whether each pair of functions are inverse functions.

8. $f(x) = x + 7$
$g(x) = x - 7$

9. $g(x) = 3x - 2$
$f(x) = \frac{x - 2}{3}$

Exercises

HOMEWORK HELP

For Exercises	See Examples
10–15	1
16–29	2
30–35	3

Find the inverse of each relation.

10. $\{(2, 6), (4, 5), (-3, -1)\}$

11. $\{(3, 8), (4, -2), (5, -3)\}$

12. $\{(7, -4), (3, 5), (-1, 4), (7, 5)\}$

13. $\{(-1, -2), (-3, -2), (-1, -4), (0, 6)\}$

14. $\{(6, 11), (-2, 7), (0, 3), (-5, 3)\}$

15. $\{(2, 8), (-6, 5), (8, 2), (5, -6)\}$

Find the inverse of each function. Then graph the function and its inverse.

16. $y = -3$

17. $g(x) = -2x$

18. $f(x) = x - 5$

19. $g(x) = x + 4$

20. $f(x) = 3x + 3$

21. $y = -2x - 1$

22. $y = \frac{1}{3}x$

23. $f(x) = \frac{5}{8}x$

24. $f(x) = \frac{1}{3}x + 4$

25. $f(x) = \frac{4}{5}x - 7$

26. $g(x) = \frac{2x + 3}{6}$

27. $f(x) = \frac{7x - 4}{8}$

GEOMETRY **The formula for the area of a circle is $A = \pi r^2$.**

28. Find the inverse of the function.

29. Use the inverse to find the radius of the circle whose area is 36 square centimeters.

Real-World Career

Meteorologist

Meteorologists use observations from ground and space, along with formulas and rules based on past weather patterns to make their forecast.

Math Online For more information, go to algebra2.com.

Determine whether each pair of functions are inverse functions.

30. $f(x) = x - 5$
$g(x) = x + 5$

31. $f(x) = 3x + 4$
$g(x) = 3x - 4$

32. $f(x) = 6x + 2$
$g(x) = x - \frac{1}{3}$

33. $g(x) = 2x + 8$
$f(x) = \frac{1}{2}x - 4$

34. $h(x) = 5x - 7$
$g(x) = \frac{1}{5}(x + 7)$

35. $g(x) = 2x + 1$
$f(x) = \frac{x - 1}{2}$

NUMBER GAMES **For Exercises 36–38, use the following information.**
Damaso asked Emilia to choose a number between 1 and 35. He told her to subtract 12 from that number, multiply by 2, add 10, and divide by 4.

36. Write an equation that models this problem.

37. Find the inverse.

38. Emilia's final number was 9. What was her original number?

TEMPERATURE **For Exercises 39 and 40, use the following information.**
A formula for converting degrees Celsius to Fahrenheit is $F(x) = \frac{9}{5}x + 32$.

39. Find the inverse $F^{-1}(x)$. Show that $F(x)$ and $F^{-1}(x)$ are inverses.

40. Explain what purpose $F^{-1}(x)$ serves.

H.O.T. Problems

EXTRA PRACTICE See pages 905 and 932.

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41. REASONING Determine the values of n for which $f(x) = x^n$ has an inverse that is a function. Assume that n is a whole number.

42. OPEN ENDED Sketch a graph of a function f that satisfies the following conditions: f does not have an inverse function, $f(x) > x$ for all x, and $f(1) > 0$.

43. CHALLENGE Give an example of a function that is its own inverse.

44. *Writing in Math* Refer to the information on page 391 to explain how inverse functions can be used in measurement conversions. Point out why it might be helpful to know the customary units if you are given metric units. Demonstrate how to convert the speed of light $c = 3.0 \times 10^8$ meters per second to miles per hour.

STANDARDIZED TEST PRACTICE

45. **ACT/SAT** Which of the following is the inverse of the function $f(x) = \frac{3x - 5}{2}$?

A $g(x) = \frac{2x + 5}{3}$ C $g(x) = 2x + 5$

B $g(x) = \frac{3x + 5}{2}$ D $g(x) = \frac{2x - 5}{3}$

46. **REVIEW** Which expression represents $f(g(x))$ if $f(x) = x^2 + 3$ and $g(x) = -x + 1$?

F $x^2 - x + 2$ H $-x^3 + x^2 - 3x + 3$

G $-x^2 - 2$ J $x^2 - 2x + 4$

Spiral Review

If $f(x) = 2x + 4$, $g(x) = x - 1$, and $h(x) = x^2$, find each value. (Lesson 7-1)

47. $f[g(2)]$ 48. $g[h(-1)]$ 49. $h[f(-3)]$

List all of the possible rational zeros of each function. (Lesson 6-9)

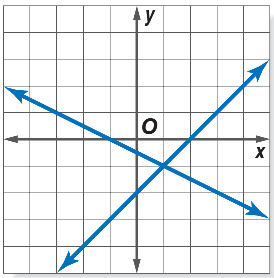
50. $f(x) = x^3 + 6x^2 - 13x - 42$ 51. $h(x) = -4x^3 - 86x^2 + 57x + 20$

Perform the indicated operations. (Lesson 4-2)

52. $\begin{bmatrix} 3 & -4 \\ 2 & 8 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 7 & 7 \\ 3 & -6 \end{bmatrix}$ 53. $\begin{bmatrix} 3 & 3 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 5 & 2 \end{bmatrix}$

54. Find the maximum and minimum values of the function $f(x, y) = 2x + 3y$ for the polygonal region with vertices at (2, 4), (−1, 3), (−3, −3), and (2, −5). (Lesson 3-4)

55. State whether the system of equations shown at the right is *consistent and independent*, *consistent and dependent*, or *inconsistent*. (Lesson 3-1)



56. **BUSINESS** The amount that a mail-order company charges for shipping and handling is given by the function $c(x) = 3 + 0.15x$, where x is the weight in pounds. Find the charge for an 8-pound order. (Lesson 2-2)

Solve each equation or inequality. Check your solutions. (Lessons 1-3, 1-4, and 1-5)

57. $2x + 7 = -3$ 58. $-5x + 6 = -4$ 59. $|x - 1| = 3$

60. $|3x + 2| = 5$ 61. $2x - 4 > 8$ 62. $-x - 3 \leq 4$

GET READY for the Next Lesson

PREREQUISITE SKILL Graph each inequality. (Lesson 2-7)

63. $y > \frac{2}{3}x - 3$ 64. $y \leq -4x + 5$ 65. $y < -x - 1$